C: The Multi-Dimensional Diffusion Equation

Consider a fixed spatial domain $\Omega \subset \mathcal{R}^3$, with $\mathbf{x} \in \Omega$, $u(\mathbf{x}, t)$ being the temperature at location $\mathbf{x}$ at time t. Let $\mathbb{D} \subset \Omega$ be any subdomain of Ω, with boundary S, and let $\nu = \nu(\mathbf{x})$ be the unit outward normal vector at $\mathbf{x} \in S$, assumed defined everywhere on surface S. Then $\phi \cdot \nu < 0$ and the outward flow of heat is negative. (This is just a sign convention.)

The heat energy in $\mathbb{D}$ is $\int_{\mathbb{D}} c\rho u d\mathbf{x}$ (the same as before in the 1D case, with c being specific heat capacity, ρ being density). The energy balance law is now
{rate of change of heat energy} = {net heat energy into $\mathbb{D}$ from the boundary per unit time} + {heat energy generated within $\mathbb{D}$ per unit time}.
This gives

$$\frac{\partial}{\partial t}\int_{\mathbb{D}} c\rho u(x,t) d\mathbf{x} = -\int_{\partial\mathbb{D}} \phi \cdot \nu \; ds + \int_{\mathbb{D}} Q(x,t) d\mathbf{x} \quad . \tag{1}$$

Applying the Divergence theorem to (1) yields

$$\frac{\partial}{\partial t}\int_{D} c\rho u d\mathbf{x} = -\int_{D} \nabla \cdot \phi dx + \int_{D} Q d\mathbf{x} \quad .$$

With c, ρ being constants, we have

$$\int \{c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q\} d\mathbf{x} = 0 \quad .$$

Since this holds for every subdomain of Ω with smooth boundary, then by a straight-forward generalization of the Lemma on page 2 of Section 3,

$$c\rho \tfrac{\partial u}{\partial t} + \nabla \cdot \phi = Q \text{ in } \Omega \quad .$$

Now, Fourier's law, which is a constitutive relation, not an actual law, states that $\phi = -k\nabla u = -k \; grad(u)$ (k = constant thermal conductivity). If we write $D := k/c\rho$, $F := Q/c\rho$, we have the multi-dimensional version of the diffusion (heat) equation

$$\frac{\partial u}{\partial t} = D\nabla^2 u + F \quad . \tag{2}$$